Upper Semicontinuous Decomposition of E^3 into Subarc's of Bing's Sling and Points

by

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A Thesis Presented to the

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This thesis is dedicated to my parents and my wife.

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" الخلاصة "

معلاق " بينق " هو منحنى بسيط مغلق معرف على الفضاء الثلاثي الأقليدى E^3 بحيث لايوجد (هوميمور فيزم) h من E^3 الى نفسه آخذه المنحنى الى دائرة . يدعى معلاق " بينق " منحنى بسيط جامع .أى قوس جزىء A من معلاق " بيق " يكون خلويا بمعنى أن أى منطقة محيطة بالقوس الجزىء A تحتوى على خلية ثلاثية يكون A بداخلها تماما .

فى هذا البحث سندرس تحلل شبه المتصل العلوى للفضاء الثلاثي E^3 الى أقواس جزيئية من معلاق " بينق " غير متقاطعة مع بعضها البعض وسوف نثبت أن هذا التحلل ينتج عنه دائما فضاءات متحلله (هو ميمورفك) الى E^3 .

فى الباب الأول سنستعرض بعض المفاهيم و النتائج من نظرية تحلل الفضاء و التى نحتاجها فى البابين اللاحقين فى الباب الثانى ندرس تركيب معلاق " بينق " وفى الباب الثالث ندرس نوعا خاصا من تحلل الفضاء E^3 المذكور سابقا .

* * *

ABSTRACT

Bing's sling is a simple closed curve in Euclidean 3-space E^3 for which there is no homeomorphism h from E^3 onto itself taking it to a circle. We say that Bing's sling is a wild simple closed curve. Any subarc A of Bing's sling is cellular that is, each neighborhood of A contains a 3-cell which contains A in its interior.

We study upper semicontinuous decompositions of Euclidean 3-space E^3 into points and pairwise disjoint subarcs of Bing's sling. We prove that such decompositions always yield decomposition spaces that are homeomorphic to E^3.

In chapter 1 we review some basic concepts and results from decomposition space theory needed in the following chapters. Chapter 2 studies the construction of Bing's sling and chapter 3 studies the special type of decomposition space of E^3 mentioned above.

CONTENTS

CHAPTER ONE

INTRODUCTION

In this chapter, we present some basic definitions and theorems in decomposition space theory which will be needed in the following chapters. We assume familiarity with basic results from general topology.

1.1 Decomposition Spaces

<u>Definition 1.1.1</u>

Let X be a topological space. Let G be a collection of pairwise disjoint non-empty subsets of X whose union is X. G is said to be a decomposition of X. We use H for the set of all non-degenerate elements in G, that is, those with more than one point.

<u>Definition 1.1.2</u>

If G is a decomposition of X, then the map P from X to G taking each x in X to the unique element $P(x)$ in G which contains x is called the projection map.

Note that in our terminology maps need not be continuous.

<u>Definition 1.1.3</u>

Let X and Y be topological spaces. A map f from X onto Y is a quotient map if a subset U of Y is open in Y iff $f^{-1}(U)$ is open in X.

Definition 1.1.4

If X is a space and G is a set and if f is a surjective map from X onto G, then the topology on G relative to which f is a quotient map is called the quotient topology. It is an easy exercise to verify the existence and uniqueness of quotient topologies.

Definition 1.1.5

Let G be a decomposition of a space X. The decomposition space or quotient space of X induced by G is the set G together with the quotient topology induced by the projection map P from X onto G. We denote this space by X/G.

Theorem 1.1.1

A closed map or an open map is a quotient map if it is surjective and continuous.

The proof follows directly from the definition.

Theorem 1.1.2

Let X/G be a decomposition space of X with projection map P. Then,

1. P is continuous.
2. If Y is a space and f from X/G to Y a function,

then f is continuous iff $f \circ P$ is continuous.

For proof of this theorem, we refer to [6, page 103].

Theorem 1.1.3

Let f be a quotient map from the space X to the space Y and let $G = \{f^{-1}(y) | y \in Y\}$. Then there is a homeomorphism Φ from X/G onto Y such that $\Phi \circ P = f$, where P is the projection map from X onto G as defined before.

For proof we refer to [6, page 106].

Theorem 1.1.4

Let X/G be a decomposition space of the space X.

1. If X is compact, then so is X/G.

2. If X is connected, then so is X/G.

3. If X is separable, then so is X/G.

4. If X is locally connected, then so is X/G.

For proof we refer to [6, page 104].

Definition 1.1.6

A decomposition G of X is said to be upper semicontinuous if for each $g \in G$ and for each open set U

containing g, $\cup\{g' \in G | g' \subset U\}$ is open in X.

<u>Theorem 1.1.5</u>

A decomposition G of a space X is upper semicontinuous iff the projection map $P: X \to X/G$ is closed.

For proof we refer to [6, page 105].

<u>Theorem 1.1.6</u>

Let G be an upper semicontinuous decomposition of a space X such that the elements of G are compact sets.

1. If X is second countable, then so is X/G.

2. If X is separable metrizable, then so is X/G.

For proof we refer to [6, page 106].

Our main objects of study are upper semicontinuous decompositions of E^3 into arcs and points. The following theorems and examples are some of the relevant previous research in this area.

<u>Theorem 1.1.7</u>

Suppose G is an upper semicontinuous decomposition of E^3 such that each g in G is cellular [see page 12 for the definition], G has only a countable number of non-degenerate elements, and their union is a G_δ set. Then the decomposition space E^3/G is homeomorphic to E^3 [Theorem 1 of [1]].

Theorem 1.1.8

Suppose G is an upper semicontinuous decomposition of E^3 such that G has only a countable number of non-degenerate elements and each is a tame arc [see page 11 for the definition]. Then E^3/G is homeomorphic to E^3. [Theorem 3 of [1]].

Example 1.1.1

There is an upper semicontinuous decomposition G of E^3 such that H consists of uncountably many tame arcs and $\cup H$ is a compact set but E^3/G is not homeomorphic to E^3. The space E^3/G is called Bing's Dogbone space. The details are in [2].

Example 1.1.2

There is an upper semicontinuous decomposition of the 3-sphere S^3 whose non-degenerate elements are all contained in a cellular arc, but the decomposition space S^3/G is not homeomorphic to S^3. Details are in [5].

We now state and prove the following two theorems which are general topology results. These theorems will be used in Chapter 3. There we will have a sequence $\langle h_i \rangle$ of homeomorphisms from E^3 onto itself which shrink nondegenerate elements into sets of arbitrarily small diameters. By applying the following two theorems to the sequence $\langle h_i \rangle$ it will be shown that E^3/G is homeomorphic to E^3.

Throughout this thesis we will use the usual metric d on E^3.

Theorem 1.1.9

Suppose that $\langle f_i \rangle$ is a sequence of homeomorphisms of E^3 onto itself. Further, suppose that there is a sequence $\langle V_i \rangle$ of open sets such that

(a) $V_i \supset V_{i+1}$,

(b) $f_{i+1} = f_i$ on $E^3 \backslash V_i$, and

(c) each component of $f_i(V_i)$ has diameter less than $1/i$.

Then $\langle f_i \rangle$ converges uniformly to a continuous function f.

Proof: For $k \geq 1$ we have $f_{i+k} = f_i$ on $E^3 \backslash V_i$, so for $p \notin V_i$, $f_{i+k}(p) = f_i(p)$. If $p \in V_i$, then let D be the component of V_i containing p. By condition (c), D is not all of E^3. Now D is open in E^3; it can not be closed in E^3, because then $E^3 \backslash D$ should be also closed and E^3 would be disconnected which is false. Hence $\bar{D} \backslash D \neq \phi$.

We have $\bar{D}\backslash D \subset E^3\backslash V_i$. So, $f_{i+k} = f_i$ on $\bar{D}\backslash D$ and therefore $f_i(\bar{D}) \cap f_{i+k}(\bar{D}) \neq \phi$ ... (1). Now for all $k \geq 1$, we have $f_i(V_i) = f_{i+k}(V_i)$. Since the f_i's are homeomorphisms and D is a component of V_i, $f_{i+k}(D)$ is a component of $f_i(V_i)$. So, there is a component D^K of V_i such that $f_i(D^K) = f_{i+k}(D)$. Then $f_i(\overline{D^K}) = f_{i+k}(\bar{D})$. From (1) we get $f_i(\overline{D^K}) \cap f_i(\bar{D}) \neq \phi$. Hence, $\overline{D^K} \cap \bar{D} \neq \phi$ ... (2).

Now $f_i(D^K)$ and $f_i(D)$ are components of $f_i(V_i)$. So, diam $f_i(D^K) < 1/i$ and diam $f_i(D) < 1/i$. Now, $f_{i+k}(p) \in f_i(D^K)$, so there is a $y \in D^K$ such that $f_i(y) = f_{i+k}(p)$. Then $d(f_i(p), f_{i+k}(p)) = d(f_i(p), f_i(y))$. Since $\overline{D^K} \cap \bar{D} \neq \phi$, we have $d(f_i(p), f_i(y)) \leq \text{diam } f_i(\overline{D^K}) + \text{diam } f_i(\bar{D}) < 1/i + 1/i = 2/i$. We conclude that $d(f_i(p), f_{i+k}(p)) < 2/i$ for all p in E^3.

For any $\varepsilon > 0$, there is an integer i with $2/i < \varepsilon/2$. Then, for $j, k \geq i$ and $p \in E^3$, $d(f_k(p), f_j(p)) \leq d(f_k(p), f_i(p)) + d(f_i(p), f_j(p)) < 2/i + 2/i < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $\langle f_i \rangle$ converges uniformly to a function f. Since the f_i's are continuous, f is also continuous.

<u>Theorem 1.1.10</u>

Suppose a sequence $\langle f_i \rangle$ of homeomorphisms of E^3 onto itself converges uniformly to a function f. Then f is onto E^3 and f is closed.

<u>Proof</u>: Let A be a closed set in E^3. We need to show that $f(A)$ is closed in E^3. To do this, let $\langle f(a_n) \rangle$ be a sequence in $f(A)$ so that $\langle f(a_n) \rangle$ converges to x_0, for some x_0 in E^3. We shall show that $x_0 \in f(A)$. Now, f_i converges to f uniformly. So, given $\varepsilon > 0$, there is N_0 such that for $i \geq N_0$, $d(f_i(a_i), x_0) < \varepsilon$. Thus, $\langle f_i(a_i) \rangle$ converges to x_0.

Now, $\exists\, m_0$ such that for $m,n \geq m_0$ we have, $d(f_m(a_m), x_0) < 1$ and $d(f_m(y), f_n(y)) < 1$ for each y. Let $i \geq m_0$. Then $d(f_i(a_i), x_0) < 1$ and $d(f_i(a_i), f_{m_0}(a_i)) < 1$, so, $d(f_{m_0}(a_i), x_0) < 2$. So, for $i \geq m_0$ we have $a_i \in f_{m_0}^{-1}(B(x_0,2))$, where $B(x_0,2)$ is the open ball with center x_0 and radius 2. But $f_{m_0}^{-1}(\overline{B(x_0,2)})$ is compact. Hence there is a subsequence $\langle a_{i_j} \rangle$ of $\langle a_i \rangle$ which converges to some x in $f_{m_0}^{-1}(\overline{B(x_0,2)})$. Since A is closed, $x \in A$. Then $\langle f(a_{i_j}) \rangle$ converges to $f(x)$, since f is continuous. But $\langle f(a_{i_j}) \rangle$ also converges to x_0, so that $f(x) = x_0$ and hence $x_0 \in f(A)$.

We now show that f is onto. Let $y \in E^3$. There is a sequence $n_1 < n_2 < n_3 < \ldots$ such that for $m, n \geq n_k$ we have $d(f_m(x), f_n(x)) < 1/k$, for each $x \in E^3$. Now suppose $k \geq j$. Then $n_k \geq n_j$ so that $d(y, f_{n_j}(f_{n_k}^{-1}(y))) = d(f_{n_k}(f_{n_k}^{-1}(y)), f_{n_j}(f_{n_k}^{-1}(y))) < 1/j$. Hence $f_{n_k}^{-1}(y) \in f_{n_j}^{-1}(B(y, 1/j))$.

Now $\overline{B(y,1/j)}$ is compact and hence so is $\overline{f_{n_j}^{-1}(B(y,1/j))} = f_{n_j}^{-1}(\overline{B(y,1/j)})$. Then the sequence $\langle f_{n_k}^{-1}(y)\rangle_{k \geq j}$ is in $\overline{f_{n_j}^{-1}(B(y,1/j))}$, for each j. Hence there is a subsequence $\langle f_{n_{k_i}}^{-1}(y)\rangle$ converging to x and $x \in \overline{f_{n_j}^{-1}(B(y,1/j))} = f_{n_j}^{-1}(\overline{B(y,1/j)})$, for each j. Hence $f_{n_j}(x) \in \overline{B(y,1/j)}$ for each j. i.e. $d(f_{n_j}(x), y) \leq 1/j$, for each j.

Now, given $\varepsilon > 0$, there exists N_0 such that $1/N_0 < \varepsilon/2$. Then for each $j \geq N_0$, $d(f_{n_j}(x), y) \leq 1/j \leq 1/N_0 < \varepsilon/2$. Also, for each $j \geq N_0$ and for each $n, n_j \geq n_{N_0}$, $d(f_n(x), f_{n_j}(x)) < 1/N_0 < \varepsilon/2$. Then $d(f_n(x),y) \leq d(f_n(x), f_{n_j}(x)) + d(f_{n_j}(x),y) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, $f_n(x)$ converges to y. But $f_n(x)$ converges to $f(x)$ since f_n converges to f uniformly. Since the limit is unique, we have $f(x) = y$. Hence, f is onto E^3.

1.2 0-dimensional Spaces

<u>Definition 1.2.1</u>

A space X has dimension 0 at a point p if p has arbitrarily small neighborhoods with empty boundaries, i.e. if for each neighborhood U of p there exists a neighborhood V of p such that $V \subset U$ and $\mathrm{Bd}V = \phi$.

Definition 1.2.2

A nonempty space X has dimension 0, written $\dim X = 0$, if X has dimension 0 at each of its points. We define the dimension of the empty set to be -1.

Remark 1.2.1

A 0-dimensional space can also be defined as a non-empty space in which there is a basis for the open sets made up of sets which are at the same time open and closed.

Example 1.2.1

Every non-empty finite or countable metrizable space X is 0-dimensional [4, page 10].

Theorem 1.2.1

A non-empty subset of a 0-dimensional space is 0-dimensional.

For proof we refer to [4, page 13].

Theorem 1.2.2

Let X be a metrizable space and M a subset of X of dimension less than or equal to 0. Suppose U_1 and U_2 are two open sets of X which cover M. Then there exist two open sets V_1 and V_2 which cover M and satisfy

$V_1 \subset U_1$, $V_2 \subset U_2$ and $V_1 \cap V_2 = \phi$.

For proof we refer to [4, page 53].

Theorem 1.2.3

Let X be a metrizable space, M a subset of X of dimension ≤ 0 and $\{U_i \mid i = 1,2, \ldots\}$ a covering of M with each U_i open in X. Then there is a covering $\{V_i \mid i = 1,2, \ldots\}$ of M such that $V_i \subset U_i$, $i=1,2,\ldots$, $V_i \cap V_j = \phi$ for $i \neq j$ and each V_i is open in X.

For proof we refer to [4, page 54].

1.3 Embeddings in E^3

Definition 1.3.1

Let M be an arc in E^3. If there is a homeomorphism Φ from E^3 onto itself such that $\Phi(M)$ is the closed unit interval $I = \{(x,y,z) \mid 0 \leq x \leq 1,\ y = z = 0\}$ in E^3, then M is said to be tamely embedded or just tame. If M is a simple closed curve in E^3 and Φ is a homeomorphism from E^3 onto itself such that $\Phi(M)$ is a polygonal simple closed curve in E^3, then M is said to be tamely embedded. In either case, if M is not tame, then it is said to be wild.

Definition 1.3.2.

A 3-cell D is a space homeomorphic to the closed unit ball $\{(x,y,z)\,|\,x^2 + y^2 + z^2 \leq 1\}$ in E^3. If Φ is a homeomorphism from D to the closed unit ball, then Int D $= \Phi^{-1}(\{(x,y,z)\,|\,x^2 + y^2 + z^2 < 1\})$. It can be shown that Int D is well-defined.

Definition 1.3.3

A subset X of E^3 is cellular if $X = \bigcap_{1}^{\infty} D_i$, where D_i is a 3-cell and Int $D_i \supset D_{i+1}$, for all i.

Proposition 1.3.1

A tame arc is cellular but the converse is not true.

Proof: Since M is tame, there is a homeomorphism Φ from E^3 onto itself such that $\Phi(M) = I$. It can be shown that I is cellular. To do this, let $D_i = [0 - \frac{1}{i}, 1 + \frac{1}{i}] \times [-\frac{1}{i}, \frac{1}{i}] \times [-\frac{1}{i}, \frac{1}{i}]$. Then D_i is a 3-cell and Int $D_i \supset D_{i+1}$ for all i. Also, $\bigcap_{i=1}^{\infty} D_i = [0,1]\times\{0\}\times\{0\} = I$. Hence I is cellular.

Now, let $\Phi^{-1}(D_i) = E_i$, for all i. Then the E_i's are 3-cells, because Φ^{-1} is a homeomorphism. Since Int $D_i \supset D_{i+1}$, Int $\Phi^{-1}(D_i) = \Phi^{-1}(\text{Int } D_i) \supset \Phi^{-1}(D_{i+1})$, so that Int $E_i \supset E_{i+1}$. Int $E_i \supset M$ for all i, as

Int $D_i \supset I$ for all i. Moreover, $\bigcap_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \Phi^{-1}(D_i) = \Phi^{-1}(\bigcap_{i=1}^{\infty} D_i) = \Phi^{-1}(I) = M$. Hence, M is cellular.

A cellular set need not be tame, for example, that each subarc of Bing's sling is cellular can be proved from the work in Chapter 3 while the fact that they are not tame is verified in [3].

CHAPTER TWO

THE CONSTRUCTION OF BING'S SLING

Bing's sling is a certain wild simple closed curve in E^3 for which any subarc is cellular. We will use subarcs of Bing's sling for the non-degenerate elements in the decompositions of E^3 that we study. The construction of Bing's sling was frist given in [3].

To construct Bing's sling we take first a solid torus T_1. A solid torus is a torus with its interior, that is, it is homeomorphic to the product of a circle and a 2-dimensional disk. Then we divide T_1 into 12 nonoverlapping cylindrical blocks $C_1, C_2, \ldots, C_{12}$, that is, each C_i is homeomorphic to the product of a line segment and a 2-dimensional disk.

In each of these blocks we put 3 solid tubes as shown in the figure below.

C_i

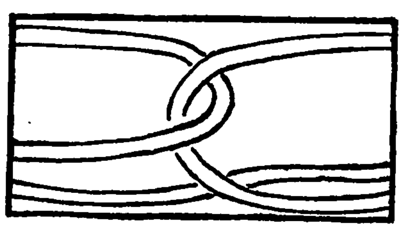
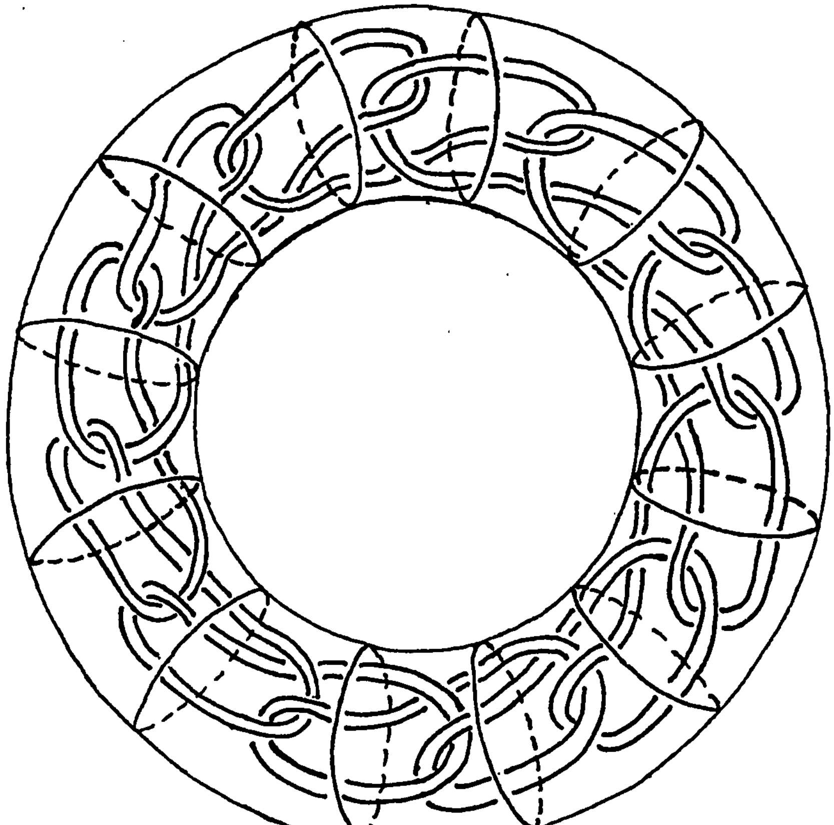


Figure 1

The tubes inside the blocks C_i fit together to make a solid torus T_2 in Int T_1. T_2 has many knots.



$T_2 \subset \mathrm{Int}\ T_1$
Figure 2

In each component of $T_2 \cap C_i$ we put 4 copies of the block with tubes. The tubes inside these smaller cylinders fit together to make a thinner solid torus T_3.

We continue this process, to get solid tori $T_1 \supset \text{Int } T_1 \supset T_2 \supset \text{Int } T_2 \supset T_3 \supset \ldots$, so that the diameters of the blocks of T_i approach 0 as i tends to ∞, and so that T_i has 12^i blocks.

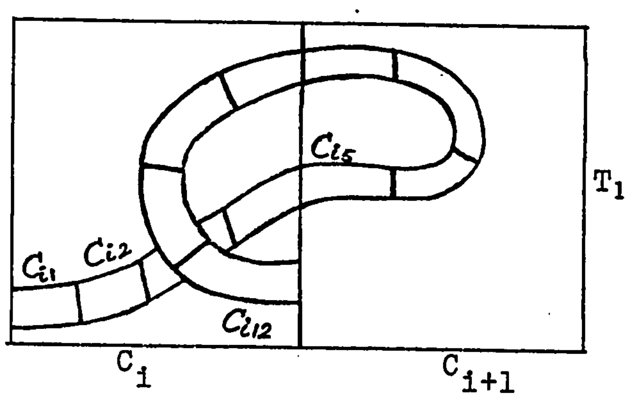


Figure 3

Now with each block C_i of T_1 we associate 12 blocks $C_{i1}, C_{i2}, \ldots, C_{i12}$ of T_2, indexed as shown. Similarly, with each block $C_{i_1 i_2 \ldots i_r}$ of T_r we associate 12 blocks $C_{i_1 i_2 \ldots ir\,1}$, $C_{i_1 i_2 \ldots i_r 2}, \ldots,$ $C_{i_1 i_2 \ldots i_r 12}$ of T_{r+1} lying in $C_{i_1 i_2 \ldots i_r}$ and the block following $C_{i_1 i_2 \ldots i_r}$ in T_r.

The following figure gives some idea about the construction of Bing's sling.

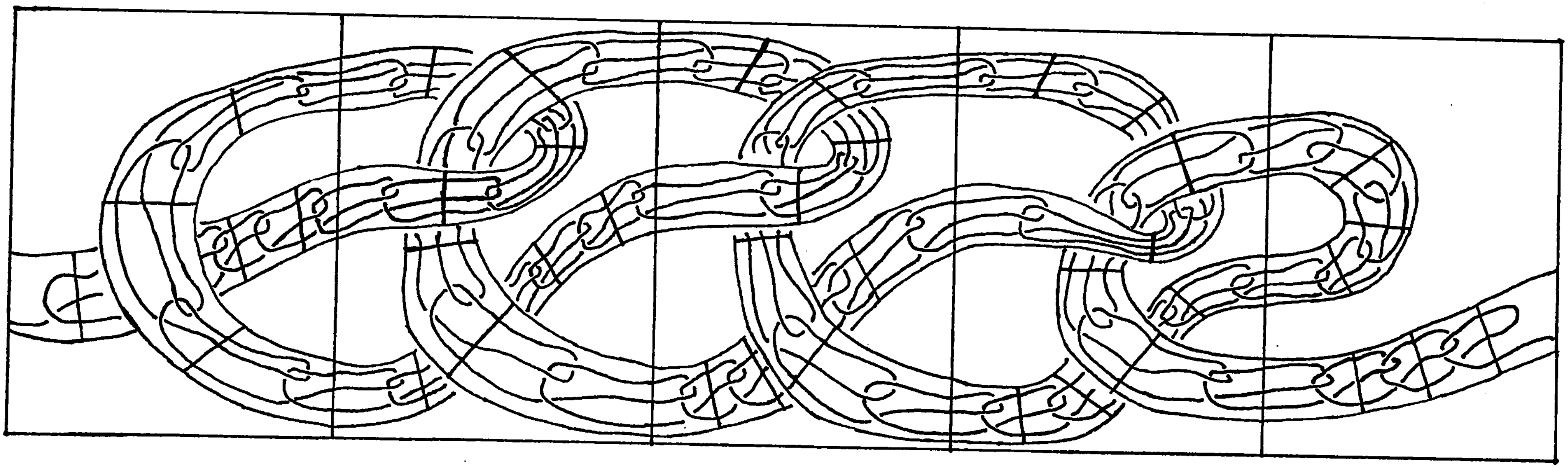


Figure 4

Let $J = \bigcap_1^\infty T_i$. Then J is a wild simple closed curve that does not pierce any disk, that is, J does not pass through any disk without intersecting the disk in more than one point. This simple closed curve is called Bing's sling.

We briefly indicate how to define a homeomorphism h from a circle $\sum$ onto $J = \bigcap_{i=1}^\infty T_i$. Let S_i be the center simple closed curve of T_i. Divide $\sum$ into 12 equal arcs $J_1, \ldots, J_{12}$ and let h_1 be a homeomorphism from $\sum$ onto S_1 taking J_i to $S_1 \cap C_i$. Then divide each J_i into 12 equal arcs. Let h_2 be a homeomorphism from $\sum$ onto S_2 taking each subsegment J_{i_j} of J_i to $C_{i_j} \cap S_2$ for $i,j = 1,2,\ldots,12$. We continue this process indefinitely and get a sequence $\langle h_i \rangle$ of homeomorphisms from $\sum$ into E^3. The sequence $\langle h_i \rangle$ converges uniformly to a homeomorphism h from $\sum$ onto J. This proves that J is a simple closed curve.

CHAPTER THREE

UPPER SEMICONTINUOUS DECOMPOSITIONS OF E^3 INTO SUBARCS OF BING'S SLING AND POINTS

In this chapter we will study upper semicontinuous decomposition spaces of E^3 where the set H of non-degenerate elements in the decomposition is a collection of pairwise disjoint subarcs of Bing's sling, so throughout this chapter G and H will have the appropriate meaning.

3.1 H is Countable

Let J be Bing's sling. J is a separable metric space, so J has a countable basis B.

Suppose H is uncountable. Now the non-degenerate elements in H are compact sets. Since each $g \in H$ is a subarc of the simple closed curve J, each $g \in H$ has interior points in J. Thus, there is a nonempty basis element B_g such that $B_g \subset g$. Since the elements of H are pairwise disjoint, $\{B_g | g \in H\}$ is a collection of pairwise disjoint sets. Hence $\{B_g | g \in H\}$ is an uncountable subset of the basis for J. This is a contradiction because the basis B for J is countable.

Hence H is countable. We write $H = \{g_i | i = 1, 2, \ldots\}$.

3.2 The Diameter of g_i Approaches Zero as i Tends to Infinity

Suppose there exists $\varepsilon > 0$ such that $\operatorname{diam} g_i \geq \varepsilon$ for infinitely many i's, say for $i_1, i_2, \ldots$. There exist a_{i_j}, b_{i_j} in g_{i_j} with $d(a_{i_j}, b_{i_j}) > \varepsilon/2$. Since J is compact, there exist subsequences $\langle a_{i_{j_k}} \rangle$, $\langle b_{i_{j_k}} \rangle$ that converge to a, b respectively. Since $d(a_{i_{j_k}}, b_{i_{j_k}}) > \varepsilon/2$, $d(a, b) \geq \varepsilon/2$, so $a \neq b$. But now the $g_{i_{j_k}}$'s are pairwise disjoint subarcs of the simple closed curve J and for any large k, $g_{i_{j_k}}$ contains a point near a and a point near b, which is impossible from the geometry of a simple closed curve.

3.3 G is an Upper Semicontinuous Decomposition of E^3

Given $g \in G$, let U be an open set in E^3 with $g \subseteq U$. Consider $A = \cup\{g' \in G | g' \cap (E^3 \backslash U) \neq \phi\}$. We want to show that A is closed, so that $E^3 \backslash A = \cup\{g' \in G | g' \subset U\}$ will be open in E^3, thus showing that G is upper semicontinuous. Any convergent sequence of points from $E^3 \backslash U$ converges to a point in $E^3 \backslash U$, but A may contain points from U.

Suppose $\langle a_n \rangle$ is a convergent sequence of points in A with $a_n \in U$. Then each a_n belongs to some non-degenerate element g_{i_n} which meets $E^3 \backslash U$. But if $\{g_{i_n} | a_n \in U\}$ were finite, then $\cup\{g_{i_n} | a_n \in U\}$ would be a closed set and the sequence $\langle a_n \rangle$ would converge to some point in $\cup\{g_{i_n} | a_n \in U\}$ which is contained in A, because $g_{i_n} \cap (E^3 \backslash U) \neq \phi$, so that $g_{i_n} \subset A$ from each i_n.

So, suppose $\{g_{i_n} | a_n \in U\}$ is countably infinite. But $\langle a_n \rangle$ converges to a_0 for some $a_0 \in E^3$. If a_0 belongs to some non-degenerate element g_{i_n} to which a_n belongs then $a_0 \in A$ since $g_{i_n} \subset A$.

Suppose a_0 does not belong to any non-degenerate element to which some a_n belongs and that $a_0 \in U$. Let $d(a_0, E^3 \backslash U) = \varepsilon > 0$. Since a_n converges to a_0, there exists n_0 such that for each $n \geq n_0$, $d(a_0, a_n) < \varepsilon/2$. But diam g_{i_n} approaches 0 as n tends to infinity. So, for $\varepsilon/2 > 0$, there exists m_0 such that for each $n \geq m_0$, diam $g_{i_n} < \varepsilon/2$. Let $N_0 = \max\{n_0, m_0\}$. Then for each $n \geq N_0$, and for each $x \in g_{i_n}$, $d(a_0, x) \leq d(a_0, a_n) + d(a_n, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, $g_{i_n} \subset U$. Thus, for each

$n \geq N_0$, $g_{i_n} \subset U$, which is a contradiction because the g_{i_n}'s meet $E^3 \backslash U$. Hence, $a_0 \in E^3 \backslash U \subset A$.

Finally, let $\langle a_n \rangle$ be a convergent sequence of points of A for which $a_i \in E^3 \backslash U$ for infinitely many i and $a_i \in U$ for infinitely many i. Now, the subsequence $\langle a_{n_i} \rangle$ with $a_{n_i} \in E^3 \backslash U$ converges to some point in $E^3 \backslash U$ since $E^3 \backslash U$ is closed. Since $\langle a_n \rangle$ is a convergent sequence, the subsequence and the sequence converge to the same point in $E^3 \backslash U \subset A$. Hence A is closed.

Consequently, E^3/G is an upper semicontinuous decomposition of E^3.

3.4 Basic Lemma

In this section we will construct homeomorphisms of E^3 onto itself which shrink each non-degenerate element of G into a set of small diameter. With this end in mind, we state and prove the following basic lemma.

Basic Lemma:

Given a positive ε and an open set U containing $\cup H$, there exists a homeomorphism h_ε of E^3 onto itself which shrinks each $g \in G$ into a set $h_\varepsilon(g)$ of diameter less than ε, is identity on $E^3 \backslash U$ and takes each component of U onto itself.

<u>Proof</u>: Let $\varepsilon > 0$ be given and let U be an open set such that $U \supset \cup H$.

Let $H_\varepsilon = \{g \in H | \text{diam } g \geq \varepsilon/2\}$. Now, H_ε is finite by the result in Section 3.2. If $H_\varepsilon = \phi$, then we need only take h_ε to be the identity, so suppose $H_\varepsilon \neq \phi$. Consider any g in H_ε. Fix a point q in $J \backslash \cup H_\varepsilon$ and let $O_g = \{x | d(x,g) < \frac{1}{2} d(g,(\cup H_\varepsilon \cup \{q\}) \backslash g)\}$. Then O_g is an open set in E^3 containing g and not meeting any other element of H_ε or the point q.

Let $V_g = \cup\{g' \in G | g' \subset O_g \cap U\}$. Then V_g is open since G is an upper semicontinuous decomposition of E^3. Further, by the definition of O_g, $\{V_g | g \in H_\varepsilon\}$ is a collection of pairwise disjoint open sets. Also, for each $g \in H_\varepsilon$, $g \subset V_g \subset U$, for any element $g' \in G$ either $g' \cap V_g = \phi$ for each $g \in H_\varepsilon$ or $g' \subset V_g$ for some $g \in H_\varepsilon$ and $q \notin V_g$ for all $g \in H_\varepsilon$. Now let us fix a g in H_ε and let r be the minimum of $\frac{1}{2} d(g, E^3 \backslash V_g)$ and $\varepsilon/4$. There exists a positive integer n_1 such that the solid torus T_{n_1} has blocks of diameter less than r.

Now, let B_1 be the union of all blocks of T_{n_1} meeting g. Then B_1 is connected since g and the blocks are all connected. $B_1 \subset V_g$, by the choice of r, so not all blocks of T_{n_1} lie in B_1 because the point q lies in J, hence in a block of T_{n_1}, but $q \notin V_g$. Thus, B_1

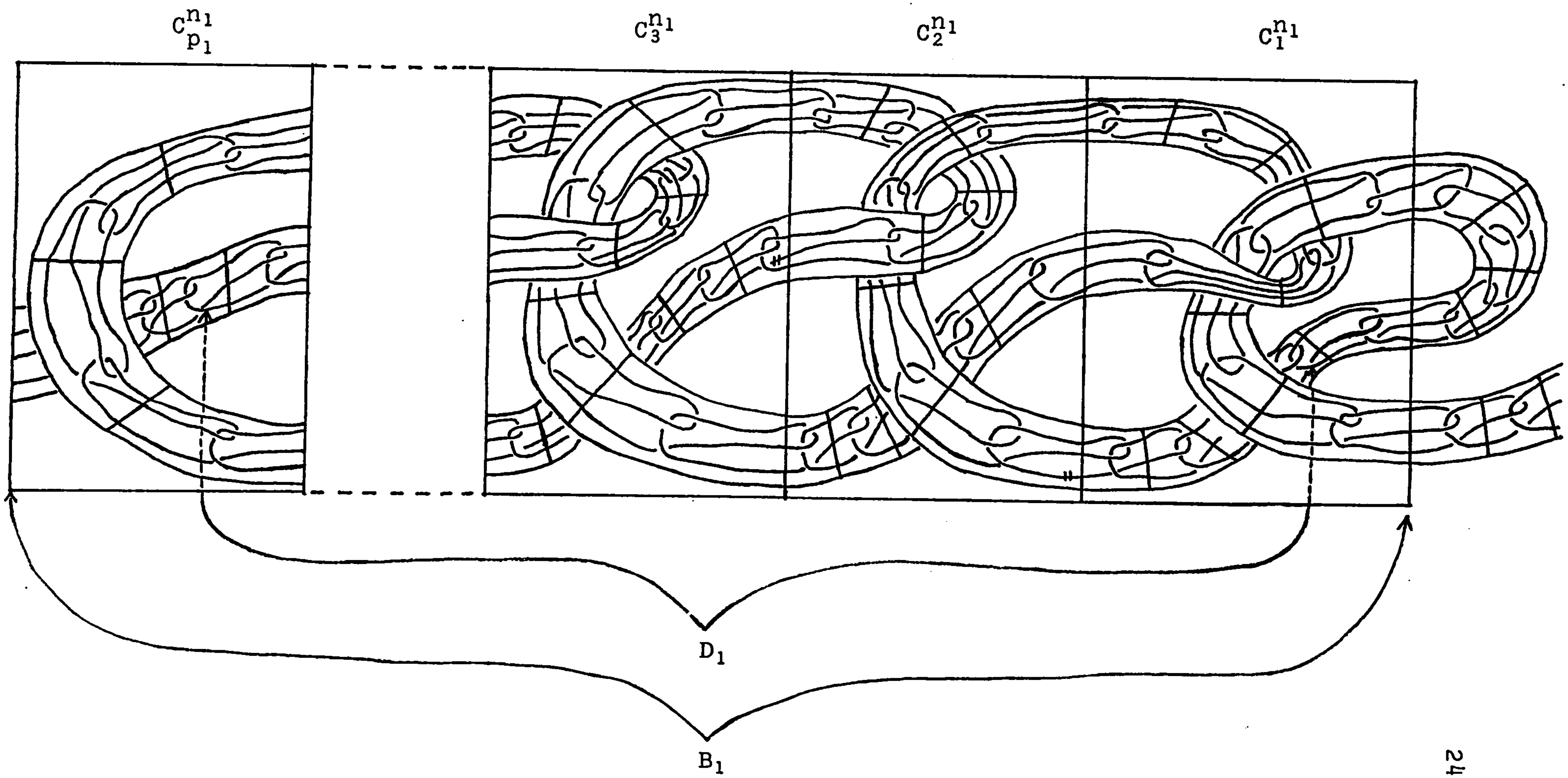


Figure 5

is a connected union of some but not all blocks of T_{n_1} and so is a 3-cell.

Now the blocks of T_{n_1} meeting g are ordered in the same way as ordered in T_{n_1}. Let them be $C_1^{n_1}, C_2^{n_1}, \ldots, C_{p_1}^{n_1}$ in this ordering, so that $B_1 = \bigcup_{i=1}^{p_1} C_i^{n_1}$.

Since $g \subset \operatorname{Int} T_{n_1}$ and every block of T_{n_1} meeting g lies in B_1, we have $g \subset \operatorname{Int} B_1 \subset B_1$. Since $B_1 \subset V_g$, the only element of G of diameter greater than or equal to $\varepsilon/2$ meeting B_1 is g itself.

Now, $g \subset \operatorname{Int} B_1$ and $\operatorname{Int} B_1$ is open. Since G is an upper semicontinuous decomposition of E^3, $W_1 = \cup\{g' \in G \mid g' \subset \operatorname{Int} B_1\}$ is open in E^3. Then $g \subset W_1 \subset \operatorname{Int} B_1$ and there exists a positive integer $m_1 > n_1$ such that each block of the solid torus T_{m_1} has diameter less than $\frac{1}{2}\, d(g, E^3 \backslash W_1)$.

As before, the union of all blocks of T_{m_1} meeting g is a 3-cell D_1. Since $m_1 > n_1$, all blocks of T_{m_1} have diameter less than $\varepsilon/4$. Also, $D_1 \subset W_1$ by properties of the diameters of blocks of T_{m_1}. Also, just as before $g \subset \operatorname{Int} D_1$. Thus $g \subset \operatorname{Int} D_1 \subset D_1 \subset W_1 \subset \operatorname{Int} B_1 \subset B_1 \subset V_g$.

Consider the 3-cell $C_1^{n_1} \cup C_2^{n_1}$. Now, let h_1 be a homeomorphism from E^3 onto itself such that $h_1 = \mathrm{Id}$ (the identity) on $E^3\setminus \mathrm{Int}(C_1^{n_1} \cup C_2^{n_1})$, h_1 takes all the blocks of D_1 meeting $C_1^{n_1}$ into the interior of $C_2^{n_1}$, and h_1 takes all blocks of D_1 lying in $C_1^{n_1} \cup C_2^{n_1}$ into $C_2^{n_1}$. The reason that such an h_1 exists is because of the special construction of the solid tori, in particular, when the sequence of blocks of T_{m_1} leaves $C_1^{n_1}$, it does not go beyond $C_2^{n_1}$ before returning. Some other blocks of T_{m_1} not lying in D_1 but contained in $C_1^{n_1}$ may be pulled into the interior of $C_2^{n_1}$ or may be stretched by h_1 so that their images meet the interior of $C_2^{n_1}$.

Thus $h_1(g) \subset \mathrm{Int}(C_2^{n_1} \cup C_3^{n_1} \cup \dots \cup C_{p_1}^{n_1})$, so that $g \subset h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup C_3^{n_1} \cup \dots \cup C_{p_1}^{n_1}))$. Because $h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup \dots \cup C_{p_1}^{n_1}))$ is open in E^3, $W_2 = \cup\{g' \in G \mid g' \subset h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup C_3^{n_1} \cup \dots \cup C_{p_1}^{n_1}))\}$ is open in E^3 and $g \subset W_2 \subset h_1^{-1}(\mathrm{Int}(C_2^{n_1} \cup \dots \cup C_{p_1}^{n_1}))$. Then there exists a positive integer $n_2 > m_1$ such that each block of the solid torus T_{n_2} has diameter less than $\frac{1}{2}\, d(g, E^3\setminus W_2)$. Since $T_{n_2} \subset \mathrm{Int}\, T_{m_1}$, the diameter of each block of T_{n_2} is less than $\varepsilon/4$.

Now, consider the 3-cell $B_2 = \bigcup_{i=1}^{p_2} h_1(C_i^{n_2})$, where $C_1^{n_2}, C_2^{n_2}, \dots, C_{p_2}^{n_2}$ are all the blocks of T_{n_2} meeting g which are ordered in the same way as ordered in T_{n_2}. Then $B_2 \subset h_1(W_2)$, because the blocks of T_{n_2} meeting g lie in W_2. Moreover, $h_1(g) \subset \mathrm{Int}\, B_2$, because

$g \subset \mathrm{Int}(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{p_2}^{n_2})$. Thus, $h_1(g) \subset \mathrm{Int}\, B_2 \subset B_2 \subset h_1(W_2) \subset \mathrm{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \subset \mathrm{Int}\, B_1 \subset B_1 \subset V_g$.

Let $W_2' = \cup\{g' \in G | g' \subset \mathrm{Int} \bigcup_{i=1}^{p_2} C_i^{n_2}\}$. Then $g \subset W_2' \subset \mathrm{Int} \bigcup_{i=1}^{p_2} C_i^{n_2}$ so $h_1(g) \subset h_1(W_2') \subset \mathrm{Int}\, B_2$. There exists a positive integer $m_2 > n_2$ such that each block of the solid torus T_{m_2} has diameter less than $\frac{1}{2} d(g, E^3 \backslash W_2')$. Since $T_{m_2} \subset \mathrm{Int}\, T_{n_2}$, all blocks of T_{m_2} have diameter less than $\varepsilon/4$.

Now, let D_2 be the image under h_1 of the union of all blocks of T_{m_2} meeting g. The interior of this union contains g and because the blocks of T_{m_2} meeting g lie in W_2', the union lies in W_2', so that $h_1(g) \subset \mathrm{Int}\, D_2 \subset D_2 \subset h_1(W_2') \subset \mathrm{Int}\, B_2 \subset B_2 \subset h_1(W_2) \subset \mathrm{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \subset \mathrm{Int}\, B_1 \subset B_1 \subset V_g$.

Consider the 3-cell $h_1(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$, where $\ell_2 < P_2$, $h_1(C_{\ell_2}^{n_2}) \subset \mathrm{Int}\, C_3^{n_1}$ and $h_1(C_{\ell_2}^{n_2} \cup C_{\ell_2+1}^{n_2} \cup \ldots \cup C_{p_2}^{n_2}) \subset \mathrm{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$. There is a homeomorphism h_2 from E^3 onto itself such that $h_2 = \mathrm{Id}$ on $E^3 \backslash \mathrm{Int}\, h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$, h_2 takes all the blocks of D_2 meeting $h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2 - 1}^{n_2})$ into the interior of $h_1(C_{\ell_2}^{n_2})$, and h_2 takes all blocks of D_2 lying in $h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$ into

$h_1(C_{\ell_2}^{n_2})$. The reason that such an h_2 exists is because of the special construction of the solid tori, in particular, when the sequence of blocks of $h_1(T_{m_2})$ leaves $h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2-1}^{n_2})$, it does not go beyond $h_1(C_{\ell_2}^{n_2})$ before returning. Some other blocks of $h_1(T_{m_2})$ not lying in D_2 but contained in $h_1(C_1^{n_2})$ may be pulled into the interior of $h_1(C_{\ell_2}^{n_2})$ or may be stretched by h_2 into $h_1(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$. Certainly, $h_2(h_1(g)) \subset \mathrm{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$.

Continuing inductively in this way, we get the following:

(1) integers $n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \ldots < n_j < m_j < \ldots < n_{p_1-2} < m_{p_1-2} < n_{p_1-1} < m_{p_1-1}$,

(2) homeomorphisms $h_1, h_2, \ldots, h_{p_1-1}$ of E^3 onto itself, and for $j = 2, 3, \ldots, p_1$ putting $\psi_{j-1} = h_{j-1} \circ h_{j-2} \circ \ldots \circ h_2 \circ h_1$,

(3) open sets $W_j = \cup\{g' \in G \mid \psi_{j-1}(g') \subset \mathrm{Int}(C_j^{n_1} \cup \ldots \cup C_{p_1}^{n_1})\}$,

(4) 3-cells $B_j = \bigcup_{i=1}^{p_j} \psi_{j-1}(C_i^{n_j})$, where $C_1^{n_j}, C_2^{n_j}, \ldots C_{p_j}^{n_j}$ are the blocks of T_{n_j} meeting g which are ordered in the same way as ordered in T_{n_j}.

(5) open sets $W_j' = \cup\{g' \in G \mid \psi_{j-1}(g') \subset \text{Int } B_j\}$,

(6) 3-cells D_j which are the image under ψ_{j-1} of the union of all blocks of T_{m_j} meeting g.

(7) 3-cells $\bigcup_{i=1}^{\ell_j} C_i^{n_j}$, where $\ell_j < P_j$.

These integers, homeomorphisms, open sets and 3-cells satisfy the following conditions:

(a) Each block of the solid torus T_{n_j} has diameter less than $\frac{1}{2}\, d(g, E^3 \backslash W_j)$,

(b) Each block of the solid torus T_{m_j} has diameter less than $\frac{1}{2}\, d(g, E^3 \backslash W_j')$,

(c) h_j takes $\psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j})$ onto itself,

(d) h_j shrinks the part of D_j meeting $\psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j - 1}^{n_j})$ into the interior of $\psi_{j-1}(C_{\ell_j}^{n_j})$,

(e) h_j takes the part of D_j lying in $\psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j})$ into $\psi_{j-1}(C_{\ell_j}^{n_j})$,

(f) $h_j = \text{Id}$ on $E^3 \backslash \text{Int } \psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j})$,

(g) $\psi_{j-1}(\bigcup_{i=1}^{\ell_j} C_i^{n_j}) \subset \text{Int}(C_j^{n_1} \cup C_{j+1}^{n_1})$,

(h) $\psi_{j-1}(C_{\ell_j}^{n_j}) \subset \text{Int } C_{j+1}^{n_1}$,

(i) $\psi_{j-1}(C_{\ell_j}^{n_j} \cup C_{\ell_j+1}^{n_j} \cup \ldots \cup C_{p_j}^{n_j}) \subset \text{Int}(C_{j+1}^{n_1} \cup C_{j+2}^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$, From the definition of the 3-cells D_j and conditions (d), (e), (h) and (i) we have $\psi_j(g) \subset \operatorname{Int}(C^{n_1}_{j+1} \cup \dots \cup C^{n_1}_{p_1})$. Thus $\psi_{p_1-1}(g) \subset \operatorname{Int}(C^{n_1}_{p_1})$ and so has diameter less than $\varepsilon/4$. Let $h_g = \psi_{p_1-1}$. Then h_g is a homeomorphism from E^3 onto itself such that $h_g = \mathrm{Id}$ on $E^3\backslash V_g$ and the diameter of $h_g(g)$ is less than $\varepsilon/4$.

Now, we want to prove that h_g maps any other g' in H meeting $B_1 \subset V_g$ into a set of diameter less than ε. So, suppose that $g' \in H$ and that g' meets $C^{n_1}_1 \cup C^{n_1}_2$ but that g' does not meet W_2. Then by definition (2) we have $h_1(g')$ does not meet $h_1(W_2)$. Now, by definitions (4) and (7) and condition (a) we have $h_1(C^{n_2}_1 \cup \dots \cup C^{n_2}_{\ell_2}) \subset h_1(W_2)$. Since, $h_1(g')$ does not meet $h_1(W_2)$, it does not meet $\operatorname{Int} h_1(C^{n_2}_1 \cup \dots \cup C^{n_2}_{\ell_2})$. Now, by condition (f) in the construction of the homeomorphism h_g, we get $h_2 = \mathrm{Id}$ on $E^3\backslash\operatorname{Int} h_1(C^{n_2}_1 \cup \dots \cup C^{n_2}_{\ell_2})$. Hence, h_2 does not move any point of $h_1(g')$. So, $h_2(h_1(g')) = h_1(g')$.

By definition (3), we have $W_3 = \cup\{g' \in G \mid h_2(h_1(g')) \subset \operatorname{Int}(C^{n_1}_3 \cup \dots \cup C^{n_1}_{p_1})\}$ and $W_2 = \cup\{g' \in G \mid h_1(g') \subset \operatorname{Int}(C^{n_1}_2 \cup \dots \cup C^{n_1}_{p_1})\}$. Consider any $g' \subset W_3$. Then $h_2(h_1(g')) \subset \operatorname{Int}(C^{n_1}_3 \cup \dots \cup C^{n_1}_{p_1})$. Suppose that

$h_1(g') \not\subset \operatorname{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$. Then $h_1(g') \cap h_1(W_2) = \phi$, by definition (3). By condition (f) we have $h_2 = \mathrm{Id}$ on $E^3\backslash\operatorname{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2})$. Again by definitions (4) and (7) and condition (a) we have $\operatorname{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2}) \subset h_1(W_2)$ so $E^3\backslash\operatorname{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{\ell_2}^{n_2}) \supset E^3\backslash h_1(W_2)$. Thus $h_2(h_1(W_2)) = h_1(W_2)$ so $h_1(g') \cap h_2(h_1(W_2)) = h_1(g') \cap h_1(W_2) = \phi$ and hence h_2 does not move any point of $h_1(g')$. Then $h_2(h_1(g')) = h_1(g') \not\subset \operatorname{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$ and thus $h_1(g') = h_2(h_1(g')) \not\subset \operatorname{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$, which is a contradiction because $g' \subset W_3$. Hence $h_1(g') \subset \operatorname{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$ and thus $g' \subset W_2$ and $W_3 \subset W_2$. So $h_2(h_1(W_3)) \subset h_2(h_1(W_2))$. Since $h_1(g') \cap h_2(h_1(W_2)) = \phi$, we have $h_1(g') \cap h_2(h_1(W_3)) = \phi$. Now, by definitions (4), (7) and condition (a) we have

$$\operatorname{Int}(h_2 \circ h_1)(C_1^{n_3} \cup \ldots \cup C_{\ell_3}^{n_3}) \subset B_3 = \bigcup_{i=1}^{P_3}(h_2 \circ h_1)(c_i^{n_3}) \subset$$

$h_2(h_1(W_3))$. Hence, $h_1(g') \cap \operatorname{Int}(h_2 \circ h_1)(C_1^{n_3} \cup \ldots \cup C_{\ell_3}^{n_3}) = \phi$. Again by condition (f) we have $h_3 = \mathrm{Id}$ on $E^3\backslash\operatorname{Int}(h_2 \circ h_1)(C_1^{n_3} \cup \ldots \cup C_{\ell_3}^{n_3})$ and thus $h_3(h_1(g')) = h_1(g')$. Similarly, $h_4, h_5, \ldots, h_{p_1-1}$ do not move any point of $h_1(g')$. Hence $h_g(g') = h_1(g')$. Since $h_1 = \mathrm{Id}$ on $E^3\backslash\operatorname{Int}(C_1^{n_1} \cup C_2^{n_1})$, $h_1(g') \subset g' \cup C_1^{n_1} \cup C_2^{n_1}$, and because g' meets $C_1^{n_1} \cup C_2^{n_1}$, $\operatorname{diam}(g' \cup C_1^{n_1} \cup C_2^{n_1}) \leq \operatorname{diam} g' + \operatorname{diam}(C_1^{n_1} \cup C_2^{n_1}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $\operatorname{diam} h_g(g') = \operatorname{diam} h_1(g') < \varepsilon$.

The construction of J shows that g is the only element of H which meets a block of T_{n_1} between $C^{n_1}_{p_1-1}$ and $C^{n_1}_{2}$. To prove this suppose $g' \neq g$ is an element in H which meets $C^{n_1}_{1}$. We want to show that g' does not meet $C^{n_1}_{3}$. So, if possible suppose that g' meets $C^{n_1}_{3}$. Now $g' \cap g = \phi$ and both g and g' are compact. Then $d(g,g') > 0$. There exists j such that each block of the solid torus T_j has diameter less than $\frac{1}{2} d(g,g')$ and thus any block of T_j meets at most one of g' and g. Since g' meets $C^{n_1}_{3}$, there exists a block C of T_j lying in $C^{n_1}_{3}$ that meets g' only. But g is connected, lies in $\bigcap_{1}^{\infty} T_i$ and so meets all the blocks between $C^{n_1}_{1}$ and $C^{n_1}_{p_1}$ of T_{n_1}. Therefore g must pass through all the blocks of T_j lying in $C^{n_1}_{3}$, in particular C. This is a contradiction. Hence g' does not meet $C^{n_1}_{3}$. Similarly, any other $g' \neq g$ in H meeting $C^{n_1}_{p_1}$ can not meet $C^{n_1}_{p_1-2}$. Hence g is the only element of H which meets a block of T_{n_1} between $C^{n_1}_{p_1-1}$ and $C^{n_1}_{2}$.

Now suppose that $g' \neq g$ in H is such that it meets $C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1}$. Then g' does not meet $C^{n_1}_{p_1-2}$. Now $g' = (g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) \cup (g' \cap (E^3 \backslash \mathrm{Int}\, B_1))$. From the construction of h_g we see that $h_g(g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) \subset C^{n_1}_{p_1}$. So, $\mathrm{diam}\, h_g(g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) \leq \mathrm{diam}\, C^{n_1}_{p_1} < \varepsilon/4$.

Since $B_1 \subset V_g$ and g' meets B_1 we have $g' \subset V_g$ and therefore $\operatorname{diam} g' < \varepsilon/2$. Hence $\operatorname{diam} h_g(g' \cap (E^3 \setminus \operatorname{Int} B_1)) < \varepsilon/2$, because $h_g = \mathrm{Id}$ on $E^3 \setminus \operatorname{Int} B_1$. Consequently, $\operatorname{diam} h_g(g') \leq \operatorname{diam} h_g(g' \cap (C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})) +$

$\operatorname{diam} h_g(g' \cap (E^3 \setminus \operatorname{Int} B_1)) \leq \varepsilon/4 + \varepsilon/2 < \varepsilon$.

Finally, from definitions (3), (4) and (7) and conditions (a) and (c) we see that if g' belongs to H and meets $C^{n_1}_1 \cup C^{n_1}_2$ and W_2 then either $g' \cap W_j = \phi$ and $g' \subset W_{j-1}$ for some j in $\{3,2,\ldots,p_1-1\}$ or $g' \subset W_j$ for all j. So we have the following two general cases, which will complete the proof that h_g takes any g' in H meeting B_1 into a set of diameter less than ε.

<u>General Case 1</u>: Suppose that g' belongs to H, that g' meets $(C^{n_1}_1 \cup C^{n_1}_2)$ and W_2, that g' does not meet W_j but that $g' \subset W_{j-1}$ for some j in $\{3,2,\ldots,p_1-1\}$. Then by definition (3) we have $\psi_{j-1}(g')$ does not meet $\psi_{j-1}(W_j)$. Now, by definitions (4) and (7) and condition (a) we have $\psi_{j-1}(C^{n_j}_1 \cup \ldots \cup C^{n_j}_{\ell_j}) \subset \psi_{j-1}(W_j)$. Since $\psi_{j-1}(g')$ does not meet $\psi_{j-1}(W_j)$, it does not meet $\operatorname{Int} \psi_{j-1}(C^{n_j}_1 \cup \ldots \cup C^{n_j}_{\ell_j})$. Now, by condition (f) in the construction of the homeomorphism h_g, we get $h_j = \mathrm{Id}$ on $E^3 \setminus \operatorname{Int} \psi_{j-1}(C^{n_j}_1 \cup \ldots \cup C^{n_j}_{\ell_j})$. Hence h_j does not

move any point of $\psi_{j-1}(g')$. So, $\psi_j(g') = h_j(\psi_{j-1}(g'))$ $= \psi_{j-1}(g')$.

By definition (3), we have $W_{j+1} = \cup\{g' \in G \mid \psi_j(g') \subset \mathrm{Int}(C^{n_1}_{j+1} \cup \ldots \cup C^{n_1}_{p_1})\}$ and $W_j = \cup\{g' \in G \mid \psi_{j-1}(g') \subset \mathrm{Int}(C^{n_1}_j \cup \ldots \cup C^{n_1}_{p_1})\}$. Consider any $g' \subset W_{j+1}$. Then $\psi_j(g') \subset \mathrm{Int}(C^{n_1}_{j+1} \cup \ldots \cup C^{n_1}_{p_1})$. Suppose that $\psi_{j-1}(g') \not\subset \mathrm{Int}(C^{n_1}_j \cup \ldots \cup C^{n_1}_{p_1})$. Then $\psi_{j-1}(g') \cap \psi_{j-1}(W_j) = \phi$, by definition (3). By condition (f) we have $h_j = \mathrm{Id}$ on $E^3 \backslash \mathrm{Int}\ \psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j})$. Again by definitions (4) and (7) and condition (a) we have $\mathrm{Int}\ \psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j}) \subset \psi_{j-1}(W_j)$ so $E^3 \backslash \mathrm{Int}\ \psi_{j-1}(C_1^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j}) \supset E^3 \backslash \psi_{j-1}(W_j)$. Thus $h_j(\psi_{j-1}(W_j)) = \psi_{j-1}(W_j)$ so $\psi_{j-1}(g') \cap h_j(\psi_{j-1}(W_j))$ $= \psi_{j-1}(g') \cap \psi_{j-1}(W_j) = \phi$ and hence h_j does not move any point of $\psi_{j-1}(g')$. Then $h_j(\psi_{j-1}(g')) = \psi_{j-1}(g') \not\subset \mathrm{Int}(C^{n_1}_j \cup \ldots \cup C^{n_1}_{p_1})$ and thus $\psi_{j-1}(g') = h_j(\psi_{j-1}(g')) \not\subset \mathrm{Int}(C^{n_1}_{j+1} \cup \ldots \cup C^{n_1}_{p_1})$, which is a contradiction because $g' \subset W_{j+1}$. Hence $\psi_{j-1}(g') \subset \mathrm{Int}(C^{n_1}_j \cup \ldots \cup C^{n_1}_{p_1})$ and thus $g' \subset W_j$ and $W_{j+1} \subset W_j$. So, $h_j(\psi_{j-1}(W_{j+1})) \subset h_j(\psi_{j-1}(W_j))$, that is $\psi_j(W_{j+1}) \subset \psi_j(W_j)$. Since $\psi_{j-1}(g') \cap \psi_j(W_j) = \phi$, we have $\psi_{j-1}(g') \cap \psi_j(W_{j+1}) =$ $= \phi$. Definitions (4) and (7) and condition (a) imply that

$$\mathrm{Int}\ \psi_j(C_1^{n_{j+1}} \cup \ldots \cup C_{\ell_{j+1}}^{n_{j+1}}) \subset B_{j+1} = \bigcup_{i=1}^{p_{j+1}} \psi_j(C_i^{n_{j+1}}) \subset$$

$\psi_j(W_{j+1})$. Hence, $\psi_{j-1}(g') \cap \text{Int}\, \psi_j(C_1^{n_{j+1}} \cup \ldots \cup C_{\ell_{j+1}}^{n_{j+1}})$ $= \phi$. Again by condition (f) we have $h_{j+1} = \text{Id}$ on $E^3 \backslash \text{Int}\, \psi_j(C_1^{n_{j+1}} \cup \ldots \cup C_{\ell_{j+1}}^{n_{j+1}})$ and thus $h_{j+1}(\psi_{j-1}(g')) =$ $\psi_{j-1}(g')$. Similarly, $h_{j+2}, h_{j+3}, \ldots, h_{p_1-1}$ do not move any point of $\psi_{j-1}(g')$. Hence $h_g(g') = \psi_{j-1}(g')$.

Now, by condition (f), for any k we have $h_k = \text{Id}$ on $E^3 \backslash \text{Int}\, \psi_{k-1}(C_1^{n_k} \cup \cdots \cup C_{\ell_k}^{n_k})$. So, condition (g) implies that h_k only moves points lying in $\text{Int}(C_k^{n_1} \cup C_{k+1}^{n_1})$. We proved before that g is the only element of H which meets a block of T_{n_1} between $C_{p_1-1}^{n_1}$ and $C_2^{n_1}$. Now, g' meets $C_1^{n_1} \cup C_2^{n_1}$ so it can not meet $C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1}$. Since $g' \subset W_{j-1}$, $\psi_{j-2}(g') \subset$ $\text{Int}(C_{j-1}^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$ by definition (3). Since $h_1 = \text{Id}$ on $E^3 \backslash C_1^{n_1} \cup C_2^{n_1}$, by conditions (f) and (g) we see that $\psi_{j-2} = h_{j-2} \circ \ldots \circ h_2 \circ h_1$ only moves points in $C_1^{n_1} \cup \ldots \cup C_{j-1}^{n_1}$ so that $\psi_{j-2}(g') \subset \text{Int}\, C_{j-1}^{n_1}$. Since h_{j-1} only moves points in $C_{j-1}^{n_1} \cup C_j^{n_1}$, we have $h_g(g')$ $= \psi_{j-1}(g') = h_{j-1}(\psi_{j-2}(g')) \subset C_{j-1}^{n_1} \cup C_j^{n_1}$. Consequently $\text{diam}\, h_g(g') \leq \text{diam}\, C_{j-1}^{n_1} + \text{diam}\, C_j^{n_1} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

General Case 2: If g' in H is such that g' meets $C_1^{n_1} \cup C_2^{n_1}$ but $\psi_{j-1}(g') \subset \text{Int}(C_j^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$ for each $j = 2,3,\ldots,p_1-1$, then $h_g(g') = \psi_{p_1-1}(g')$ and $\psi_{p_1-2}(g') \subset$

$\mathrm{Int}(C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})$. Thus $h_{p_1-1}(\psi_{p_1-2}(g')) \subset \mathrm{Int}(C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})$. Hence $h_g(g') = \psi_{p_1-1}(g') \subset \mathrm{Int}(C^{n_1}_{p_1-1} \cup C^{n_1}_{p_1})$. Consequently $\mathrm{diam}\, h_g(g') \leq \mathrm{diam}\, C^{n_1}_{p_1-1} + \mathrm{diam}\, C^{n_1}_{p_1} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$.

Since H_ε is finite, we let $H_\varepsilon = \{g_1, g_2, \ldots, g_r\}$. Now, let $h_\varepsilon = h_{g_1} \circ h_{g_2} \circ \ldots \circ h_{g_r}$. Since B_1 is connected and contained in U it lies in the component of U containing g. Then using the conditions (f) and (g) we see that h_j maps B_1 onto itself and therefore any component of the open set U onto itself. Using this property in the construction of h_g and the definition of h_ε, it follows that h_ε maps any component of the open set U onto itself.

Consider any $g' \in G$. If g' meets an open set V_{g_i} for some $g_i \in H_\varepsilon$, then $g' \subset V_{g_i}$. For $j \neq i$, $h_{g_j} = \mathrm{Id}$ on V_{g_i} because $h_{g_j} = \mathrm{Id}$ on $E^3 \backslash V_{g_j}$ and $V_{g_i} \cap V_{g_j} = \phi$, so $h_\varepsilon(g') = h_{g_i}(g')$. Since h_{g_i} shrinks each g' in V_{g_i} into a set of diameter less than ε, we have $\mathrm{diam}\, h_{g_i}(g') < \varepsilon$ and hence $\mathrm{diam}\, h_\varepsilon(g') < \varepsilon$.

If $g' \cap V_{g_j} = \phi$, for all $g_j \in H_\varepsilon$ then since $h_{g_j} = \mathrm{Id}$ on $E^3 \backslash V_{g_j}$, we have $h_\varepsilon(g') = (h_{g_1} \circ \ldots \circ h_{g_r})(g') = g'$. Hence, $\mathrm{diam}\, h_\varepsilon(g') = \mathrm{diam}\, g' < \varepsilon/2$, because each element of G with diameter $\geq \varepsilon/2$ lies in some V_{g_i}.

Since for each j, $V_{g_j} \subset U$, $h_{g_j} = \text{Id}$ on $E^3\backslash U$ for each j. Hence $h_\varepsilon = \text{Id}$ on $E^3\backslash U$. Consequently, h_ε is a homeomorphism from E^3 onto itself which shrinks each $g' \in G$ into a set $h_\varepsilon(g')$ of diameter less than ε, is Id on $E^3\backslash U$ and takes each component of U onto itself. This completes the proof of the basic lemma.

3.5 E^3/G is Homeomorphic to E^3

In this section we will prove the following.

Main Theorem. Suppose G is a decomposition of E^3 such that the non-degenerate elements of G are pairwise disjoint subarcs of Bing's sling. Then the decomposition space E^3/G is homeomorphic to E^3.

The theorem is proved by repeated applications of the Basic Lemma in Section 3.4. We must define a continuous function h from E^3 onto itself such that $G = \{h^{-1}(x) | x \in E^3\}$ and h is a quotient function. Then by Theorem 1.1.3 we will prove that E^3/G is homeomorphic to E^3. We will obtain h as the uniform limit of a sequence $\langle h_i \rangle$ of homeomorphisms from E^3 onto itself.

In Section 3.1 we proved that the set H of non-degenerate elements is countable and in Section 3.3 we proved that G is an upper semicontinuous decomposition of E^3. We use these results in the following proof of the above theorem.

<u>Proof</u>. Let us consider $\varepsilon = 1$ and open set $U = E^3$. Then by the Basic Lemma in Section 3.4, there is a homeomorphism h_1 of E^3 onto itself which takes each $g' \in G$ into a set $h_1(g')$ of diameter less than 1.

Before we describe $h_2, h_3, \ldots,$ we shall describe some open sets in the decomposition space E^3/G. Since H is countable and E^3/G is a metric space by Theorem 1.1.6, we have by Example 1.2.1, that H is 0-dimensional in the decomposition space E^3/G. For each i, let us consider open balls $B(g_i, 1/2m)$ in E^3/G, where $m = 1, 2, 3, \ldots$. Then, $\operatorname{diam} B(g_i, 1/2m) \leq 1/m$ and $g_i \in B(g_i, 1/2m)$, so that $\{B(g_i, 1/2m) \mid i = 1,2,\ldots\}$ is a countable open covering for the 0-dimensional space H in E^3/G. Then by Theorem 1.2.3 for each m there exists a collection of open sets $\{M_i^m \mid i = 1,2,\ldots\}$ covering H such that $M_i^m \subset B(g_i, 1/2m)$ for each i, and $M_i^m \cap M_j^m = \phi$ for $i \neq j$. Thus, the diameter of M_i^m is less than or equal to $1/m$. Also $\bigcup_{i=1}^{\infty} M_i^m \subset \bigcup_{i=1}^{\infty} B(g_i, 1/2m) = B(\{g_i \mid i = 1,2,\ldots\}, 1/2m)$.

Since in E^3 we have $\overline{\bigcup_{i=1}^{\infty} g_i} \subset J$, the subset $\overline{\bigcup_{i=1}^{\infty} g_i}$ is a compact subset of E^3. Now, $P(\overline{\bigcup_{i=1}^{\infty} g_i}) = \overline{\{g_i | i = 1,2,\ldots\}}$, because P is closed by Theorem 1.1.5. Since P is continuous, $\overline{\{g_i | i = 1,2,\ldots\}}$ is compact in E^3/G. Now, there is a neighborhood N_m of $\bigcup_{i=1}^{\infty} g_i$ in E^3 such that $\overline{N}_m$ is compact in E^3 and therefore $P(\overline{N}_m)$ is compact in E^3/G with $\{g_i | i = 1,2,\ldots\} \subset P(N_m) \subset \bigcup_{i=1}^{\infty} M_i^m$. Let $V_m = P(N_m)$. Since $P^{-1}(V_m) = P^{-1}P(N_m) = N_m$, we have $P(N_m) = V_m$ is open in E^3/G because P is a quotient map. Then $\overline{V}_m = \overline{P(N_m)} = P(\overline{N}_m)$. Since $P(\overline{N}_m)$ is compact, $\overline{V}_m$ is compact in E^3/G. Thus, V_m is a bounded open set in the decomposition space E^3/G such that $\bigcup_{i=1}^{\infty} M_i^m \supset V_m \supset H$. Since the M_i^m's are pairwise disjoint open sets, a component of V_m must be a subset of some M_j^m. Hence each component of V_m has diameter less than or equal to $1/m$.

Thus, we get a sequence of open sets $V_1, V_2, V_3, \ldots$, in the decomposition space E^3/G such that for each m, $V_m \supset H$, each component of V_m is of diameter less than or equal to $1/m$ in E^3/G and $\overline{V}_m$ is compact.

Let $V_m^* = \cup\{g' \in G | g' \in V_m\}$. Thus $V_m^* \subset E^3$. Now, for any $g_i \in H$, diam $h_1(g_i) < 1$ and so there exists an open set $W_i \supset h_1(g_i)$ such that diam $W_i < 1$. By upper semi-continuity of G, $O_i = \cup\{g' \in G | g' \subset h_1^{-1}(W_i)\}$ is open in

E^3 and $g_i \subset O_i \subset h_1^{-1}(W_i)$. Since $P^{-1}P(O_i) = O_i$, $\{P(O_i) | i = 1,2,\ldots\}$ is an open cover of H in E^3/G, so by 0-dimensionality of H in E^3/G and Theorem 1.2.3, there exists an open cover $\{C_i\}$ of H in E^3/G such that the C_i's are pairwise disjoint and $C_i \subset P(O_i)$ for each i. Let $U_1 = (\bigcup_{i=1}^{\infty} P^{-1}(C_i)) \cap V_1^*$. Then U_1 is open in E^3.

Now, if K is a component of $h_1(U_1)$, then $h_1^{-1}(K)$ is a component of U_1. Since the $P^{-1}(C_i)$'s are pairwise disjoint, $h_1^{-1}(K)$ must lie in some $P^{-1}(C_i)$, hence in $P^{-1}P(O_i) = O_i \subset h_1^{-1}(W_i)$, so that $K \subset W_i$, and hence diam $K \leq$ diam $W_i < 1$.

Now, $V_1 = P(N_1)$ and $P^{-1}(V_1) = P^{-1}P(N_1) = N_1$. Then $\overline{P^{-1}(V_1)} = \overline{N}_1$. Since $\overline{N}_1$ is compact in E^3, we have $\overline{P^{-1}(V_1)}$ is compact in E^3. But $V_1^* = P^{-1}(V_1)$, so $\overline{V_1^*} = \overline{P^{-1}(V_1)}$. Hence $\overline{V^*}_1$ is compact in E^3. Since $\overline{U}_1 \subset \overline{V_1^*}$, it is compact in E^3. Hence, h_1 is uniformly continuous on U_1. Let $\delta > 0$ correspond to $\varepsilon = 1/2$ in the definition of the uniform continuity of h_1 on U_1. Then, for the given δ, Basic Lemma says there is a homeomorphism ψ_δ of E^3 onto itself which shrinks each g' in G into a set of diameter less than δ and which is the identity on $E^3 \setminus U_1$. Consider $h_2 = h_1 \circ \psi_\delta$. Then $h_2 = h_1$ on $E^3 \setminus U_1$

and $\operatorname{diam} h_2(g') = \operatorname{diam} h_1(\psi_\delta(g')) < 1/2$ for each $g' \in G$, by the uniform continuity of h_1 on U_1.

Thus, h_2 is a homeomorphism from E^3 onto itself which shrinks each g' in G into a set of diameter less than $1/2$ and $h_2 = h_1$ on $E^3\backslash U_1$.

Just as we constructed U_1 and h_2, we can inductively construct homeomorphisms $h_3, h_4, \ldots,$ and open sets U_2 $U_3, U_4, \ldots,$ so that h_i shrinks each g' in G into a set $h_i(g')$ with $\operatorname{diam} h_i(g') < 1/i$, $h_{i+1} = h_i$ on $E^3\backslash U_i$, $U_{i+1} \subset U_i \cap V^*_{i+1}$ and each component of $h_i(U_i)$ has diameter less than $1/i$.

Thus, $\langle h_i\rangle$ is a sequence of homeomorphisms of E^3 onto itself. Further, there is a sequence $\langle U_i\rangle$ of open sets such that (a) $U_i \supset U_{i+1}$, (b) $h_{i+1} = h_i$ on $E^3\backslash U_i$ and (c) each component of $h_i(U_i)$ has diameter less than $1/i$. Hence, by Theorem 1.1.9 $\langle h_i\rangle$ converges uniformly to a continuous function h. Also by Theorem 1.1.10, h is onto E^3 and h is closed. Since h is closed, it is a quotient map by Theorem 1.1.1.

Now, we want to prove that $G = \{h^{-1}(x) \mid x \in E^3\}$. Suppose p, q belong to g', for some g' in H. Then for each i, g' is contained in a component of the open set U_i so that $h_i(g')$ is contained in a component of the open set $h_i(U_i)$ and hence for each i, $d(h_i(p), h_i(q)) < 1/i$.

Since h_i converges uniformly to h, it also converges pointwise to h. Then, $h_i(p)$ converges to $h(p)$, $h_i(q)$ converges to $h(q)$ as $i \to \infty$ and therefore $d(h(p), h(q)) \leq 0$. So, $d(h(p), h(q)) = 0$. Hence $h(p) = h(q)$.

Now, suppose p and q belong to different elements $P(p)$ and $P(q)$ of G. E^3/G is a metric space and $d(P(p), P(q)) > 0$. There is a positive integer m such that $\frac{1}{m} < \frac{1}{3} d(P(p), P(q))$. Consider the subset V_m of G. Now $V_m \supset H$ and each component of V_m has diameter less than or equal to $1/m$. Hence $P(p)$ and $P(q)$ do not belong to the same component of $V_m \cup \{P(p)\} \cup \{P(q)\}$ in the decomposition space E^3/G. Hence $P(p)$ and $P(q)$ are not subsets of the same component of $U_m \cup \{p\} \cup \{q\}$ in E^3.

Let D_p and D_q be the components of $U_m \cup \{p\} \cup \{q\}$ such that $P(p) \subset D_p$ and $P(q) \subset D_q$. Then $P(D_p)$ and $P(D_q)$ will be contained in some components of $V_m \cup \{P(p)\} \cup \{P(q)\}$. Since $d(P(p), P(q)) > 3/m$ and components of $V_m \cup \{P(p)\} \cup \{P(q)\}$ have diameters less than or equal to $1/m$, $d(P(D_p), P(D_q))$ must be at least $1/m$. Hence $\overline{P(D_p)} \cap \overline{P(D_q)} = \phi$, so $P^{-1}\overline{P(D_p)} \cap P^{-1}\overline{P(D_q)} = \phi$. But

$\overline{D}_p \subset P^{-1}\overline{P(D_p)}$, and $\overline{D}_q \subset P^{-1}\overline{P(D_q)}$ so that $\overline{D}_p \cap \overline{D}_q = \phi$.

Now, $h_m(p) \in h_m(D_p) \subset h_m(U_m \cup \{p\} \cup \{q\})$ and $h_m(q) \in h_m(D_q) \subset h_m(U_m \cup \{p\} \cup \{q\})$. Also, by the construction of the h_i's we see that $h_m, h_{m+1}, h_{m+2}, \ldots,$ all map D_p onto $h_m(D_p)$ and D_q onto $h_m(D_q)$. Thus, $h_m(D_p) = h_{m+1}(D_p) = \ldots = h_{m+k}(D_p)$ and $h_m(D_q) = h_{m+1}(D_q) = \ldots = h_{m+k}(D_q)$. So, $h_{m+k}(p) \in h_m(D_p)$ and $h_{m+k}(q) \in h_m(D_q)$ for all k. Thus, $h(p) \in \overline{h_m(D_p)}$ and $h(q) \in \overline{h_m(D_q)}$. Hence, $h(p) \neq h(q)$ as $\overline{D}_p \cap \overline{D}_q = \phi$. Consequently, $G = \{h^{-1}(x) | x \in E^3\}$. Since h is a quotient map, by Theorem 1.1.3 there is a homeomorphism from E^3/G onto E^3. Hence, the decomposition space E^3/G is homeomorphic to E^3.

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